

SELF-REGULATION IN THE BOLKER-PACALA MODEL

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ABSTRACT. The Markov dynamics is studied of an infinite system of point entities placed in \mathbb{R}^d , in which the constituents disperse and die, also due to competition. Assuming that the dispersal and competition kernels are continuous and integrable we show that the evolution of states of this model preserves their sub-Poissonicity, and hence the local self-regulation (suppression of clustering) takes place. Upper bounds for the correlation functions of all orders are also obtained for both long and short dispersals, and for all values of the intrinsic mortality rate.

1. INTRODUCTION

An actual task of applied mathematics is the description of the dynamics of large systems of living entities, see [1, 11, 12]. This relates, in particular, to the systems in which the dynamics amounts to the appearance (birth) and disappearance (death) of the constituents. The disappearance of a given entity caused by its interaction with the rest of the community is interpreted as the result of *competition*.

In the simplest birth-and-death models, the state space is $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then the only observed result of the trade-off between the appearance and disappearance is the dynamics of the number of entities in the population. The theory of such models goes back to A. Kolmogorov and W. Feller, see [2, Chapter XVII]. In this theory, the time evolution of the probability of having n entities is obtained from the Kolmogorov equation with a tridiagonal infinite matrix containing birth and death rates. An important generalization here is to place the entities in a continuous habitat, \mathcal{H} , usually a subset of \mathbb{R}^d , $d \geq 1$. Among the advantages of this approach is the possibility to study the system at both local and global levels, where the local structure is determined by the interaction between the entities dependent on their spatial positions. A paramount task of this study is to describe how does the local structure of a given system affects its global behavior. Typically, the entities interacting with a given entity lie in a compact subset of \mathcal{H} . If \mathcal{H} itself is compact, the qualitative difference between the global and the local is inessential. Hence, to see the difference between them one has to place the system into a noncompact habitat. A finite birth-and-death system in a noncompact habitat always occupies its compact subset and has a tendency to disperse to the empty parts by placing there the newborn entities. It can thus be classified as a *developing* system in which the possibly existing interactions have little influence on the global behavior, see [10]. Therefore, the importance of the local competition in determining the global behavior of a birth-and-death system can fully be understood if the system is *developed*, i.e., is infinite and placed in a noncompact habitat.

In this work, we continue, cf. [4, 5, 6, 10], dealing with the model introduced in [1, 11], often called Bolker-Pacala or Bolker-Pacala-Dieckmann-Law model. Here the habitat is the Euclidean space \mathbb{R}^d . The phase space is the set Γ of all locally finite subsets $\gamma \subset \mathbb{R}^d$, i.e., such that $\gamma_\Lambda := \gamma \cap \Lambda$ is finite whenever $\Lambda \subset \mathbb{R}^d$ is compact. For a compact Λ , one defines the counting map $\Gamma \ni \gamma \mapsto |\gamma_\Lambda|$ ($|\cdot|$ denotes cardinality). Then Γ is equipped with the σ -field $\mathcal{B}(\Gamma)$ generated by all $\Gamma^{\Lambda,n} := \{\gamma \in \Gamma : |\gamma_\Lambda| = n\}$, $n \in \mathbb{N}_0$ and Λ compact. This allows for considering probability measures on Γ as states of the system, including Poisson

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states in which the entities are independently distributed over \mathbb{R}^d . For the homogeneous Poisson state π_\varkappa with density $\varkappa > 0$ and every compact Λ , one has

$$\pi_\varkappa(\Gamma^{\Lambda,n}) = (\varkappa V(\Lambda))^n \exp(-\varkappa V(\Lambda)) / n!, \quad n \in \mathbb{N}_0, \quad (1.1)$$

where $V(\Lambda)$ denotes Lebesgue's measure (volume) of Λ . A state μ can be called *sub-Poissonian* if for each compact $\Lambda \subset \mathbb{R}^d$, the following holds

$$\forall n \in \mathbb{N}_0 \quad \mu(\Gamma^{\Lambda,n}) \leq C_\Lambda \varkappa_\Lambda^n / n!, \quad (1.2)$$

with some positive constants C_Λ and \varkappa_Λ . Thus, the sub-Poissonian states are characterized by the lack of *heavy tails* or *clustering*. The entities in such a state are either independent in taking their positions or 'prefer' to stay away of each other. The set of *finite* configurations $\Gamma_0 := \cup_{n \in \mathbb{N}_0} \{\gamma \in \Gamma : |\gamma| = n\}$ is clearly measurable. In a state with the property $\mu(\Gamma_0) = 1$, the system is (μ -almost surely) *finite*. In this note, we consider infinite systems and hence deal with states μ such that $\mu(\Gamma_0) = 0$.

In dealing with states on Γ one employs *observables* – appropriate functions $F : \Gamma \rightarrow \mathbb{R}$. Their evolution is obtained from the Kolmogorov equation

$$\frac{d}{dt} F_t = L F_t, \quad F_t|_{t=0} = F_0, \quad t > 0, \quad (1.3)$$

where the generator L specifies the model. The states' evolution is then obtained from the Fokker–Planck equation

$$\frac{d}{dt} \mu_t = L^* \mu_t, \quad \mu_t|_{t=0} = \mu_0, \quad (1.4)$$

related to that in (1.3) by the duality $\mu_t(F_0) = \mu_0(F_t) := \int_\Gamma F_t(\gamma) \mu_0(d\gamma)$. The model discussed in this work is specified by the following

$$\begin{aligned} (L F)(\gamma) &= \sum_{x \in \gamma} E^-(x, \gamma \setminus x) [F(\gamma \setminus x) - F(\gamma)] \\ &+ \int_{\mathbb{R}^d} E^+(x, \gamma) [F(\gamma \cup x) - F(\gamma)] dx, \end{aligned} \quad (1.5)$$

where $E^+(x, \gamma)$ and $E^-(x, \gamma)$ are state-dependent birth and death rates, respectively. We take them in the following forms

$$E^+(x, \gamma) = \sum_{y \in \gamma} a^+(x - y), \quad (1.6)$$

$$E^-(x, \gamma) = m + \sum_{y \in \gamma} a^-(x - y), \quad (1.7)$$

where $a^+ \geq 0$ and $a^- \geq 0$ are the *dispersal* and *competition kernels*, respectively, $m \geq 0$ is the intrinsic mortality rate. This model plays a significant role in the mathematical theory of ecological systems, see [12]. Its recent study can be found in [4, 5, 6]. The particular case of (1.5), (1.7) with $a^- \equiv 0$ is the *continuum contact model*, see [9] and the references therein. In this work, we aim at understanding the ecological consequences of the competition presented in (1.5).

Remark 1.1. *For the kernels a^\pm , one has the following possibilities:*

- (i) (short dispersal) *there exists $\theta > 0$ such that $a^-(x) \geq \theta a^+(x)$ for all $x \in \mathbb{R}^d$;*
- (ii) (long dispersal) *for each $\theta > 0$, there exists $x \in \mathbb{R}^d$ such that $a^-(x) < \theta a^+(x)$.*

In case (i), a^+ decays faster than a^- , and hence each daughter entity competes with her mother. Such models are usually employed to describe the dynamics of cell communities, see [3]. An instance of the short dispersal is given by a^+ with finite range, i.e., $a^+(x) \equiv 0$ for all $|x| \geq r$, and $a^-(x) > 0$ for such x . In case (ii), a^- decays faster than a^+ , and hence some of the offsprings can be out of reach of their parents. Models of this kind can be

adequate, e.g., in plant ecology with the long-range dispersal of seeds. In this study, the model parameters are supposed to satisfy the following.

Assumption 1.2. *The kernels a^\pm in (1.6) and (1.7) are continuous and belong to $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. According to this we set $\langle a^\pm \rangle = \int_{\mathbb{R}^d} a^\pm(x) dx$ and $\|a^\pm\| = \sup_{x \in \mathbb{R}^d} a^\pm(x)$.*

Like in [5, 6], the evolution of states will be described by means of correlation functions. To explain the essence of this approach let us consider the set of all compactly supported continuous functions $\theta : \mathbb{R}^d \rightarrow (-1, 0]$. For a state, μ , its *Bogoliubov* functional is

$$B_\mu(\theta) = \int_\Gamma \prod_{x \in \gamma} (1 + \theta(x)) \mu(d\gamma), \quad (1.8)$$

with θ running through the mentioned set of functions. For the homogeneous Poisson measure π_\varkappa , it takes the form

$$B_{\pi_\varkappa}(\theta) = \exp \left(\varkappa \int_{\mathbb{R}^d} \theta(x) dx \right).$$

Having this in mind we will consider those states μ for which the functional (1.8) can be written down in the form

$$B_\mu(\theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} k_\mu^{(n)}(x_1, \dots, x_n) \theta(x_1) \cdots \theta(x_n) dx_1 \cdots dx_n, \quad (1.9)$$

where $k_\mu^{(n)}$ is a symmetric element of $L^\infty((\mathbb{R}^d)^n)$. It is the *n-th order correlation function* of μ . In the contact model, for each $t > 0$ and $n \in \mathbb{N}$ the correlation functions satisfy the following estimates

$$\text{const} \cdot n! c_t^n \leq k_t^{(n)}(x_1, \dots, x_n) \leq \text{const} \cdot n! C_t^n. \quad (1.10)$$

Thus, the corresponding state does not satisfy (1.2), and hence the clustering does occur in this model. In view of this, the main question arising here is whether the competition contained in L can suppress clustering. In such a case, one can say that the *local self-regulation* takes place in this model. The answer given Theorems 2.1 and 2.2 below is in affirmative.

2. THE RESULTS

By $\mathcal{B}(\mathbb{R})$ and $\mathcal{P}(\Gamma)$ we denote the sets of all Borel subsets of \mathbb{R} and the set of all probability measures on $(\Gamma, \mathcal{B}(\Gamma))$, respectively. By definition, the subset $\mathcal{P}_{\text{exp}}(\Gamma) \subset \mathcal{P}(\Gamma)$ consists of all those μ for which B_μ can be continued, as a function of θ , to an exponential type entire function on $L^1(\mathbb{R}^d)$. It can be shown that a given μ belongs to $\mathcal{P}_{\text{exp}}(\Gamma)$ if and only if B_μ can be written down as in (1.9) where the correlation functions satisfy

$$\|k_\mu^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} \leq C \exp(\vartheta n), \quad n \in \mathbb{N}_0, \quad (2.1)$$

with some $C > 0$ and $\vartheta \in \mathbb{R}$. In other words, $k_\mu^{(n)}$ satisfies the *Ruelle bound*, see [8, Section 6]. In view of (2.1), each $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ satisfies (1.2) and hence is sub-Poissonian.

A function $G : \Gamma_0 \subset \Gamma \rightarrow \mathbb{R}$ is $\mathcal{B}(\Gamma)/\mathcal{B}(\mathbb{R})$ -measurable, see [5], if and only if, for each $n \in \mathbb{N}$, there exists a symmetric Borel function $G^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ such that

$$G(\eta) = G^{(n)}(x_1, \dots, x_n), \quad \text{for } \eta = \{x_1, \dots, x_n\}. \quad (2.2)$$

Like in (2.2), we introduce $k_\mu : \Gamma_0 \rightarrow \mathbb{R}$ such that $k_\mu(\eta) = k_\mu^{(n)}(x_1, \dots, x_n)$ for $\eta = \{x_1, \dots, x_n\}$, $n \in \mathbb{N}$. We also set $k_\mu(\emptyset) = 1$. Then we pass from (1.4) to the corresponding Cauchy problem for the correlation functions

$$\frac{d}{dt} k_t = L^\Delta k_t, \quad k_t|_{t=0} = k_{\mu_0}. \quad (2.3)$$

In non-equilibrium statistical mechanics, the corresponding problem is known as the *BBGKY hierarchy*. The action of L^Δ presents as follows, cf. [4, 5, 6],

$$\begin{aligned} (L^\Delta k)(\eta) &= (L^{\Delta,-}k)(\eta) + \sum_{x \in \eta} E^+(x, \eta \setminus x) k(\eta \setminus x) \\ &+ \int_{\mathbb{R}^d} \sum_{x \in \eta} a^+(x - y) k(\eta \setminus x \cup y) dy, \end{aligned} \quad (2.4)$$

where

$$(L^{\Delta,-}k)(\eta) := -E^-(\eta)k(\eta) - \int_{\mathbb{R}^d} \left(\sum_{y \in \eta} a^-(x - y) \right) k(\eta \cup x) dx, \quad (2.5)$$

and

$$E^-(\eta) := \sum_{x \in \eta} E^-(x, \eta \setminus x) = m|\eta| + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^-(x - y). \quad (2.6)$$

By (2.1) it follows that $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ implies $|k_\mu(\eta)| \leq C \exp(\vartheta|\eta|)$, holding for λ -almost all $\eta \in \Gamma_0$, some $C > 0$, and $\vartheta \in \mathbb{R}$. In view of this, we set

$$\mathcal{K}_\vartheta := \{k : \Gamma_0 \rightarrow \mathbb{R} : \|k\|_\vartheta < \infty\}, \quad (2.7)$$

where $\|k\|_\vartheta = \text{ess sup}_{\eta \in \Gamma_0} \{|k_\mu(\eta)| \exp(-\vartheta|\eta|)\}$. Clearly, (2.7) defines a Banach space. In the following, we use the ascending scale of such spaces \mathcal{K}_ϑ , $\vartheta \in \mathbb{R}$, with the property $\mathcal{K}_\vartheta \hookrightarrow \mathcal{K}_{\vartheta'}$ for $\vartheta < \vartheta'$. Here \hookrightarrow denotes continuous embedding. Then $\mathcal{K} := \cup_{\vartheta \in \mathbb{R}} \mathcal{K}_\vartheta$ is equipped with the corresponding inductive topology that turns it into a locally convex space.

For each $\vartheta \in \mathbb{R}$ and $\vartheta' > \vartheta$, the expressions in (2.4), (2.5) and (2.6) can be used to define the corresponding bounded linear operators $L_{\vartheta', \vartheta}^\Delta$ acting from \mathcal{K}_ϑ to $\mathcal{K}_{\vartheta'}$. Their operator norms can be estimated similarly as in [6, eqs. (3.11), (3.13)], which yields

$$\|L_{\vartheta', \vartheta}^\Delta\| \leq \frac{4(\|a^+\| + \|a^-\|)}{e^2(\vartheta' - \vartheta)^2} + \frac{\langle a^+ \rangle + m + \langle a^- \rangle e^{\vartheta'}}{e(\vartheta' - \vartheta)}. \quad (2.8)$$

By means of the collection $\{L_{\vartheta', \vartheta}^\Delta\}$ with all $\vartheta \in \mathbb{R}$ and $\vartheta' > \vartheta$ we introduce the corresponding continuous linear operators acting on \mathcal{K} , and thus define the Cauchy problems (2.3) in this space. By the (global in time) solutions we mean continuously differentiable functions $[0, +\infty) \ni t \mapsto k_t \in \mathcal{K}$ such that both equalities in (2.3) hold.

Theorem 2.1. *Under Assumption 1.2 the following holds: For each $\mu_0 \in \mathcal{P}_{\text{exp}}(\Gamma)$, the problem in (2.3) with $L^\Delta : \mathcal{K} \rightarrow \mathcal{K}$ as in (2.4) – (2.6) and (2.8) has a unique solution k_t such that, for each $t > 0$, there exists a unique state $\mu_t \in \mathcal{P}_{\text{exp}}(\Gamma)$ for which $k_t = k_{\mu_t}$.*

Theorem 2.2. *Let ϑ_0 be such that $k_{\mu_0} \in \mathcal{K}_{\vartheta_0}$. Then, for all $t \geq 0$, the mentioned above solution k_t , corresponding to this k_{μ_0} , satisfies the following estimates:*

- (i) Case $\langle a^+ \rangle > 0$ and $m \in [0, \langle a^+ \rangle]$: for each $\delta < m$ (long dispersal) or $\delta \leq m$ (short dispersal), there exists a positive C_δ such that $\log C_\delta \geq \vartheta_0$ and

$$k_t(\eta) \leq C_\delta^{|\eta|} \exp\left((\langle a^+ \rangle - \delta)|\eta|t\right), \quad \eta \in \Gamma_0.$$

- (ii) Case $\langle a^+ \rangle > 0$ and $m > \langle a^+ \rangle$: for each $\varepsilon \in (0, m - \langle a^+ \rangle)$, there exists a positive C_ε such that $\log C_\varepsilon \geq \vartheta_0$ and

$$k_t(\eta) \leq C_\varepsilon^{|\eta|} \exp(-\varepsilon t), \quad \eta \neq \emptyset. \quad (2.9)$$

- (iii) Case $\langle a^+ \rangle = 0$:

$$k_t(\eta) \leq k_0(\eta) \exp[-E^-(\eta)t], \quad \eta \in \Gamma_0. \quad (2.10)$$

If $m = 0$ and $a^-(x) = \theta a^+(x)$, then

$$k_t(\eta) = \theta^{-|\eta|}, \quad t \geq 0, \quad (2.11)$$

is a stationary solution.

The proof of these statements is based on the following

Lemma 2.3. *Let a^\pm satisfy Assumption 1.2. Then one finds $b \geq 0$ and $\theta > 0$ such that*

$$b|\eta| + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^-(x - y) \geq \theta \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^+(x - y), \quad (2.12)$$

holding for all $\eta \in \Gamma_0$.

The proof of the lemma (quite technical) can be found in [7].

3. COMMENTS AND COMPARISON

The condition of continuity of a^- in Assumption 1.2 can be relaxed. In fact, it is enough to assume that a^- is measurable and separated away from zero in some ball. For a^+ , it is enough to have a continuous $\tilde{a}^+ \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ such that $\tilde{a}^+(x) \geq a^+(x)$ for almost all x .

By Theorem 2.1, adding competition to the continuum contact model, cf. (1.10), yields the local self-regulation – no matter how long the dispersal is. In the short dispersal case, the inequality in (2.12) readily holds with $b = 0$. Then the most intriguing question here is whether it can hold in the long dispersal case. In [6, Proposition 3.7], it was shown that measurable a^+ and a^- satisfy (2.12) with some b and θ if $a^-(x)$ is separated away from zero for $|x| < r$ with some $r > 0$, and $a^+(x) \equiv 0$ for $|x| \geq R$ with some $R > 0$ with the possibility $R > r$. Another choice of a^+ and a^- satisfying (2.12) can be, see [6, Proposition 3.8],

$$a^\pm(x) = \frac{c_\pm}{(2\pi\sigma_\pm^2)^{d/2}} \exp\left(-\frac{1}{2\sigma_\pm^2}|x|^2\right),$$

with all possible $c_\pm > 0$ and $\sigma_\pm > 0$. An important example of a^\pm which both Propositions 3.7 and 3.8 of [6] do not cover is the case where a^- has finite range and a^+ is Gaussian as above. The novelty of our present – rather unexpected – result is that (2.12) is satisfied for *any* a^+ and a^- as in Assumption 1.2, and hence the local self-regulation is achieved by applying *any kind of competition*.

Now let us compare our results with those of [4, 5]. In [4], the model was supposed to satisfy the conditions, see [4, Eqs. (3.38) and (3.39)], which can be formulated as follows: (a) condition (i) in Remark 1.1 holds with a given $\theta > 0$; (b) $m > 16\langle a^- \rangle / \theta$ for this θ . Then the global evolution $k_0 \mapsto k_t$ was obtained in \mathcal{K}_ϑ with some $\vartheta \in \mathbb{R}$ by means of a C_0 -semigroup. No information was available on whether k_t is a correlation function and hence on the sign of k_t . In [5], the restrictions were relaxed to imposing the short dispersal condition. Then the evolution $k_0 \mapsto k_t$ was obtained in a scale of Banach spaces \mathcal{K}_α as in Theorem 2.1, but on a bounded time interval. Like in [4], also here no information was obtained on whether k_t is a correlation function.

Theorem 2.2 gives a complete characterization of the evolution $k_0 \mapsto k_t$. For $m < \langle a^+ \rangle$ (short dispersal) or $m \leq \langle a^+ \rangle$ (long dispersal), the evolution described in Theorem 2.1 takes place in an ascending scale $\{\mathcal{K}_{\vartheta_t}\}_{t \geq 0}$ of Banach spaces. If $m > \langle a^+ \rangle$, the evolution holds in one and the same space. The only difference between the cases of long and short dispersals is that one can take $\delta = m$ in the latter case. This yields different results for $m = \langle a^+ \rangle$, where the evolution takes place in the same space \mathcal{K}_ϑ with $\vartheta = \log C_m$. Note also that for $m = 0$, one should take $\delta < 0$. For $m > \langle a^+ \rangle$, it follows from (2.9) that the population dies out: for $\langle a^+ \rangle > 0$, the following holds

$$k_{\mu_t}^{(n)}(x_1, \dots, x_n) \leq e^{-\varepsilon t} k_{\mu_0}^{(n)}(x_1, \dots, x_n), \quad t > 0,$$

for some $\varepsilon \in (0, m - \langle a^+ \rangle)$, almost all (x_1, \dots, x_n) , and each $n \in \mathbb{N}$. For $m > 0$ and $\langle a^+ \rangle = 0$, by (2.10) we get

$$k_{\mu_t}^{(n)}(x_1, \dots, x_n) \leq \exp(-nmt) k_{\mu_0}^{(n)}(x_1, \dots, x_n), \quad t > 0.$$

This means that $k_{\mu_t}^{(n)}(x_1, \dots, x_n) \rightarrow 0$ as $n \rightarrow +\infty$ for sufficiently big $t > 0$. This phenomenon does not follow from (2.9). Finally, we mention that (2.11) corresponds to a special case of short dispersal. Until the present work no results on the extinction as in (2.9) and on the case of $a^+ \equiv 0$ were known.

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